THE EIGHTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Eighth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Friday and Saturday, December 27–28, 1901. A single day's session now often barely suffices for the adequate presentation of the programme of an ordinary meeting of the Society. It was therefore decided last year to extend the annual meeting to cover two days, in the hope of relieving considerably the severe compression occasioned by the constantly increasing number of papers presented. The energetic response of the Society's productive forces has, however, been such as to leave the problem precisely where it was before. The reading of twenty-seven or more mathematical papers in two days is, it appears, a possible feat, but hardly one for sober commendation. It is becoming a serious question whether it will not be necessary in the near future to adopt a practice of selection, permitting the presentation, even then in rather condensed form, of more important papers only, those of lesser interest or value being read in severely brief abstract or by title. This plan would possess certain advantages in point of dignity over the present hurried procedure; it would also permit better opportunity for intelligent discussion of important tendencies and larger interests. The annual meeting and the summer meeting stand in especial need of some such relief.

During the past year, the Society has in all respects maintained a vigorous growth. The membership has increased from 357 to 375, the number of papers read from 112 to 136. The total attendance of members at all meetings was 231, and the number attending at least one meeting was 131. The Council has been enlarged and an assistant secretary has been added to the administrative forces. The presidential term has been extended to two years, and the date of the presidential address has been changed to the more appropriate occasion of the last annual meeting of the presidential term. An especially important event was the deposit of the library of the Society in the charge of Columbia University, by which the books become generally accessible to our members. A catalogue will be issued and rules for the use of the library will be formulated as soon as prac-

ticable. Professor Pomeroy Ladue, who has faithfully discharged the duties of Librarian of the Society since 1895, has resigned this position, being succeeded by Professor D. E. Smith. Resolutions expressing appreciation of Professor Ladue's long and valuable services were adopted by the Council.

The attendance of members at the annual meeting was fifty-nine, slightly surpassing the highest previous record of all meetings and marking an increase of fifty per cent. over any previous annual meeting. The following members were recorded as present:

Dr. Grace Andrews, Mr. C. H. Ashton, Dr. G. A. Bliss, Professor Maxime Bôcher, Professor Joseph Bowden, Professor E. W. Brown, Dr. J. E. Clarke, Professor F. N. Cole, Professor E. S. Crawley, Dr. W. S. Dennett, Dr. William Findlay, Professor G. E. Fisher, Professor T. S. Fiske, Mr. W. B. Ford, Dr. A. S. Gale, Professor Fanny C. Gates, Dr. G. B. Germann, Dr. W. A. Granville, Professor Harris Hancock, Professor James Harkness, Professor J. N. Hart, Professor Edwin Haviland, Jr., Dr. E. R. Hedrick, Dr. A. A. Himowich, Dr. E. V. Huntington, Professor Harold Jacoby, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Professor Pomeroy Ladue, Professor W. W. Landis, Professor Gustave Legras, Dr. C. T. Lewis, Dr. Emory McClintock, Dr. Isabel Maddison, Dr. Emily N. Martin, Professor Mansfield Merriman, Professor E. H. Moore, Professor F. Morley, Dr. J. L. Patterson, Mr. J. C. Pfister, Professor James Pierpont, Dr. I. E. Rabinovitch, Professor J. K. Rees, Professor E. D. Roe, Dr. Arthur Schultze, Professor I. J. Schwatt, Professor Charlotte A. Scott, Dr. S. E. Slocum, Professor P. F. Smith, Dr. Virgil Snyder, Dr. H. F. Stecker, Dr. W. M. Strong, Professor J. H. Van Amringe, Professor E. B. Van Vleck, Professor L. C. Wait, Professor A. G. Webster, Professor J. B. Webb, Professor R. S. Woodward.

Morning and afternoon sessions were held on each day. The President of the Society, Professor Eliakim Hastings Moore, occupied the chair, being relieved at the later sessions by Vice-President Professor Thomas S. Fiske. The Council announced the election of the following persons to membership in the Society: Professor R. E. Allardice, Stanford University; Dr. Grace Andrews, Columbia University; Mr. S. E. Brasefield, Michigan State Agricultural College; Mr. W. E. Brooke, University of Minnesota; Professor T. C. Esty, University of Rochester; Mr. L. L. Jackson, State Normal School, Brockport, N. Y. Seven applications for

admission to the Society were received. Reports were received from the Secretary, Treasurer, Librarian, and Auditing Committee. These reports will appear in the Annual Register.

An enjoyable social feature of the meeting was the dinner held on Friday evening at Hotel Marlborough. Falling between the two days of the meeting, this was a remarkably successful occasion. Fifty persons were present, including several representatives of the American Physical Society, which met on Friday.

At the annual election, which was held on Saturday morning, the following officers and members of the Council were chosen:

Vice-Presidents,	Professor MAXIME BÔCHER, Professor FRANK MORLEY.
Secretary,	Professor F. N. COLE.
Treasurer,	Dr. W. S. DENNETT.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.

Members of the Council to serve until December, 1904,

Professor POMEROY LADUE, Professor P. F. SMITH,
Professor G. A. MILLER, Professor E. B. VAN VLECK.

The following papers were presented at the meeting:

(1) Dr. VIRGIL SNYDER: "Further types of unicursal sextic scrolls."

(2) Dr. EMORY MCCLINTOCK: "On the nature and use of the functions employed in the recognition of quadratic residues."

(3) Professor HAROLD JACOBY: "A theorem concerning the method of least squares."

(4) Professor HARRIS HANCOCK: "The theory of maxima and minima of functions of several variables."

(5) Professor Sir R. S. BALL: "Recent researches in the theory of screws."

(6) Dr. H. F. STECKER: "On surfaces whose geodetic lines are represented by curves of the second degree when represented conformally upon the plane."

(7) Professor CHARLOTTE A. SCOTT: "A recent method for treating the intersections of plane curves."

(8) Dr. EDWARD KASNER: "Two principles in the theory of multiple forms."

(9) Mr. D. R. CURTISS: "On the invariants of a homogeneous quadratic differential equation of the second order."

(10) Professor ARNOLD EMCH: "Some applications of the theory of assemblages."

(11) Professor G. A. MILLER: "On a method for constructing all the groups of order p^m."

(12) Dr. S. E. SLOCUM: "Note on the transformation of a group into its canonical form."

(13) Dr. E. R. HEDRICK: "On the characteristics of differential equations."

(14) Professor CHARLOTTE A. SCOTT: "On the circuits of plane curves."

(15) Professor F. MORLEY and Mr. A. B. COBLE: "On plane quartic curves."

(16) Professor MAXIME BÔCHER: "On the real solutions of systems of two homogeneous linear differential equations of the first order."

(17) Professor E. H. MOORE: "On the projective axioms of geometry."

(18) Dr. E. R. HEDRICK: "Remarks on the sufficient conditions in the calculus of variations."

(19) Dr. L. P. EISENHART: "Note on isotropic congruences."

(20) Dr. L. P. EISENHART: "Lines of length zero on surfaces."

(21) Dr. W. B. FITE: "Concerning the class of a group of order p^m that contains an operator of order p^{m-2} or p^{m-3}, p being a prime."

(22) Dr. EDWARD KASNER: "A characteristic property of the parabolic curve of the nth order."

(23) Dr. CARL GUNDERSEN: "On the content or measure of assemblages of points."

(24) Mr. J. W. YOUNG: "On the holomorphisms of a group."

(25) Professor P. F. SMITH: "On the resolution of orthogonal transformations."

(26) Mr. SAUL EPSTEEN: "Proof that the group of an irreducible linear differential equation is transitive."

(27) Dr. W. B. FORD: "On the uniform convergence of Fourier's series."

Professor Ball was introduced by Professor E. W. Brown,

Mr. Curtiss by Professor Bôcher, Dr. Gunderson by Professor Fiske. Mr. Young's paper was presented to the Society through Dr. Stecker, Dr. Epsteen's through Professor Lovett. In the absence of the authors, the papers of Dr. Fite and Mr. Young were read by Dr. Stecker, and the papers of Professor Jacoby, Professor Emch, Professor Miller, Dr. Eisenhart, Professor Smith, and Dr. Epsteen were read by title. Abstracts of the papers follow below.

Dr. Snyder's paper considered sextic scrolls as generated by lines joining corresponding points of two unicursal curves, the sum of whose orders is six. Either or both directing curves may be replaced by a multiple curve of lower order, or by curves of higher order which intersect in one or more self-corresponding points. Different types exist according to the configuration of the multiple points and of the assumed points which define the correspondence. This list together with that presented at the Ithaca meeting, comprises 66 types. The only further types are those which may arise from limiting cases of these, or from imaginary nodal curves. Neither of these forms have been considered.

The integer n is or is not a quadratic residue of the prime k according as the value of Legendre's function $(n/k) \equiv n^{\frac{1}{2}(k-1)} \pmod{k}$ is $+1$ or -1. The evaluation of (n/k) is performed by depressing the numerical value of n, if greater than k, and by clearing it of powers of 2, so as to make n odd and less than k. The "law of quadratic reciprocity" is then employed to transpose the numbers, and the new number on the left hand is depressed as before, and so on till the result is attained. Dr. McClintock has discovered simple relations affecting the right hand number so as to permit the depression of the two numbers alternately, without transposition. His paper treats of this class of functions in the most comprehensive manner, developing their various relations first from one definition and then from another, and closes with practical rules and illustrations. It has appeared in the January number of the *Transactions*.

Professor Jacoby's paper is in abstract as follows: Let there be given two series of observation equations as follows:

$$\begin{aligned} a_1x + b_1y + c_1z + \cdots + n_1 &= 0, \\ a_2n + b_2y + c_2z + \cdots + n_2 &= 0, \\ &\cdots\cdots\cdots\cdots \end{aligned} \qquad (1)$$

$$\begin{aligned} a_1x + b_1y + c_1z + \cdots + p_1w + \cdots + n_1 &= 0, \\ a_2x + b_2y + c_2z + \cdots + p_2w + \ldots + n_2 &= 0, \\ \cdots\cdots\cdots\

In Professor Hancock's paper the treatment is divided into two heads: 1° where no subsidiary conditions involving the variables are present; 2° where there are subsidiary conditions. The functions are supposed to be regular within the realms considered and may therefore be developed in convergent power series. The first necessary condition for the existence of maximal or minimal values is the vanishing of the first derivatives. The second condition is made to depend upon homogeneous quadratic forms. These forms must be definite and may vanish: 1° when all the variables vanish; 2° for other values of the variables. The discussion as regards the sign of these forms is then made to depend upon the quotients of determinants of different orders and upon the reality and signs of the roots of symmetrical determinants. The question of the existence of a maximum or minimum is considered, the asymptotic approach to an upper or lower limit, etc. The theory is illustrated with numerous examples and applications.

Professor Sir R. S. Ball's paper is in abstract as follows: The movement of a material system of the most general type can be represented with the help of a geometrical entity termed a screw chain, associated with a metric quantity possessing the dimensions of an angular velocity. Any small displacement of a system free or constrained in any way can be represented by a twist about a screw chain. Any set of forces acting on such a system can be represented by a wrench on a screw chain. The meaning of reciprocal screw chain systems is explained. The problem of impulsive forces was considered. It was shown that the number of principal screw chains of inertia equals the number of degrees of freedom of the system. A quiescent system receiving an impulsive wrench on a principal screw chain of inertia will commence to move by twisting about that screw chain. The differential equation which must be satisfied by the function expressing the kinetic energy of a system when screw chain coördinates are employed was obtained. The physical meaning of this equation was considered, and the meaning of permanent screw chains. A generalization of Euler's dynamical equation followed.

Dr. Stecker's paper considers the determination, in detail, of surfaces whose geodetic lines are represented by curves of the second degree in the plane. There are six different types of resulting differential equations, and six corresponding systems of partial differential equations. For three of these

it is shown that no solution exists. For a fourth case the results are found to be

$$E = \frac{1}{[A(v)u^2 + B(v)u + C(v)]^2},$$

$$G = \frac{[D(v)u + H(v)]^2}{[A(v)u^2 + B(v)u + C(v)]^2} + \frac{\Psi(v)}{A(v)u^2 + B(v)u + C(v)},$$

$$F = \frac{D(v)u + H(v)}{[A(v)u^2 + B(v)u + C(v)]^2},$$

$$K = A(v)\,C(v) - \tfrac{1}{4}B^2(v),$$

where K is the Gaussian curvature of the surface. For the fifth type

$$E = \frac{1}{\lambda c_1 v^2 + c_2}, \quad G = \frac{c_1 v^2}{[\lambda c_1 v^2 + c_2]^2}, \quad F = 0,$$

$$K = \tfrac{3}{2}\frac{(c_1 v - \lambda)}{v^2(\lambda c_1 v^2 + c_2)} + \lambda^2 c_1,$$

and λ depends upon the eccentricity of the curve. The sixth case comes from the fifth by an interchange of u and v and a change of certain constants.

A recent memoir by Dr. F. S. Macaulay (*Proceedings of the London Mathematical Society*, volume 31) throws an interesting light on the nature of the conditions to which the coefficients in the equation of a curve must be subject in order that the equation may be expressible in the form $Pu + Qv = 0$. The conditions can be written as a single "prime equation" with all equations obtained from it by a particular process of derivation. The nature of the set of equations thus obtained is investigated by Miss Scott more minutely than in Dr. Macaulay's memoir, with the result that different and simpler proofs are obtained for the theorems; the fundamental theorem in these proofs is that if two equations have a common derivative, then they have a common source for any two curves for which the two equations are satisfied.

In a paper presented at the summer meeting Dr. Kasner considered the relation between the theories of double binary forms with respect to cogredient and digredient transformations. In the present paper the author extends the principle involved to: 1° multiple binary forms, *i. e.*, forms involving any number k of binary variables, and 2°

double n-ary forms, *i. e.*, forms involving two sets of n-ary variables. The results are as follows: The cogredient theory of any system of forms 1° is, with certain restrictions, equivalent to the digredient theory of an enlarged system formed by adjoining k binary bilinear forms. The contragredient theory of any system of forms 2° is, with certain restrictions, equivalent to the digredient theory of the same system after adjoining a certain bilinear form.

Using Lie's theory, Mr. Curtiss investigates those functions of the coefficients and their derivatives, for the equation

$$(y'')^2 + 4p_2(y')^2 + p_4y^2 + 4p_3yy' + 2q_2yy'' + 4p_2y'y'' = 0,$$

which remain invariant under the transformation

$$y = \lambda(x)\cdot\eta, \quad \xi = \mu(x),$$

λ and μ being arbitrary functions of x. The dependent variable alone is first transformed and a complete system of seminvariants obtained; *i. e.*, a system whose members are functionally independent of each other and on which all other seminvariants must functionally depend. The invariants under the general transformation are then determined in terms of this system of seminvariants, a complete system of these also being obtained. The paper concludes with some applications connected with subgroups of the general transformation.

The notion of the length of any continuous line is obtained by the method of displacements contained in the theory of continuous groups. To measure the length of a limited portion of a line, or of a closed line, Euclid's algorithm, resulting in a continued fraction

$$a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \cdots,$$

is applied. In this manner the theory of irrational numbers as established by Cantor, Dedekind, and others is immediately applicable to measurable quantities. Studying the open polygon of an infinite number of finite equal sides inscribed in a circle of length u, Professor Emch is led to the theorem: Every real number $u > 1$ may be represented by a series of the form

$$u = \cfrac{1}{\dfrac{1}{n_1} - \dfrac{1}{n_1 n_2} + \dfrac{1}{n_1 n_2 n_3} - \dfrac{1}{n_1 n_2 n_3 n_4} + \cdots},$$

where $n_1 < n_2 < n_3 < n_4 < \cdots$ are positive integers. For example,

$$\pi = \cfrac{1}{\dfrac{1}{3} - \dfrac{1}{3.22} + \dfrac{1}{3.22.118} - \cdots}.$$

For $u < 1$, we have

$$u = \frac{1}{n_1} - \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} - \frac{1}{n_1 n_2 n_3 n_4} + \cdots.$$

A group G of order p^m (p being any prime) contains at least one invariant subgroup H of order p^{m-1}. If G is represented as a regular substitution group, then H has p systems of intransitivity and its p transitive constituents (H_1, H_2, $\cdots$, H_p) are transformed cyclically by every substitution of G which is not contained in H. The object of Professor Miller's paper is to find a practical method to determine the substitutions which must be added to H in order to obtain all the groups of order p^m that include H, even when H is non-abelian. The special case when H is abelian was considered by the author in the *Transactions*, volume 2, page 264. In the present paper the same general lines of thought are pursued, but the results are considerably more comprehensive. The paper will be offered for publication to the *American Journal of Mathematics*.

Dr. Slocum's paper is in abstract as follows: If we have given the finite equations defining a family of transformations T_a (with continuous parameters $a_1, \cdots, a_r$) of a given r-parameter group G, then by the introduction of new parameters $\mu_1, \cdots, \mu_r$ these equations can be transformed into the canonical equations of the group by the method given by Lie in his Transformationsgruppen, volume 3, pp. 609–611. In the canonical form these equations will define a family of transformations T_μ of G. But to a definite transformation of the family T_a with finite parameters $a_1, \cdots, a_r$ and generated by an infinitesimal transformation of G, there may correspond a transformation of the family T_μ, which has at least one of its parameters $\mu_1, \cdots, \mu_r$ infinite in all branches, and which therefore can not be generated by an

infinitesimal transformation of G. It is the purpose of the paper to illustrate this by an example, and also to point out the error in Lie's demonstration, whereby this exception arises.

The first paper of Dr. Hedrick deals with the characteristics of differential equations. The ordinary discussion of the subject is entirely laid aside, and characteristic curves are defined and developed (first for partial differential equations of the first order) as the curves for which the equation cannot be reduced to normal form, *i. e.*, for which the Cauchy-Kowalewski existence proofs cannot be carried out. Merely on this assumption the ordinary equations for characteristic curves and strips are produced; and it is shown that the curves and strips so defined possess the properties ordinarily attributed to characteristics. The further course of the theory would then be as ordinarily given, and is not dwelt upon in the paper. In this connection a proof is given that every partial differential equation of the first order has non-analytic solutions.

The possible families of characteristic curves are then considered, and it is shown that a given three-parameter family of curves is not, in general, the family of characteristic curves of any partial differential equation of the first order.

Finally, a connection is established between the theory of characteristic curves and the calculus of variations, it being shown that the characteristic curves of any equation are the extremals of a certain simple problem of the calculus of variations. The same general topics are then discussed with reference to equations of the second order, and the analogous results are obtained.

The paper, which follows to a large extent recent lectures by Professor Hilbert in Göttingen, will be offered for publication to the *Annals of Mathematics*.

The substance of Miss Scott's second paper is a proof that for any even degree there exist curves consisting of a single circuit without visible singularities of any kind. This proof is obtained by means of the fact that a curve $f=0$ cannot have a real inflexion within a region in which

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

is positive.

Clifford (Works, page 117) asserts that the general plane quartic a_x^4 possesses a contravariant curve a_ξ^4 of class 4, and of degree 5 in the coefficients of a, such that taking any line conic b, if the polar of b as to a is β, then the polar of β as to a is b. This assertion is not true in general. The conditions under which it is true are obtained by Professor Morley and Mr. Coble in algebraic form and interpreted geometrically. There is. however, in general for two plane curves, one of order 4 and one of class 4, a "double-six" of conics say b_i and $\beta_\varkappa$ $(i, \varkappa = 1, \cdots, 6)$ such that b_i is apolar to $\beta_\varkappa$ when $i \neq \varkappa$; b_i and β_i have the polar property stated by Clifford. From this point of view (compare Hilbert, "Zur binären Invariantentheorie," *Mathematische Annalen*, volume 28, page 383) the quartic is attacked.

Professor Bôcher established for a system of two homogeneous linear differential equations of the first order theorems analogous to those given by Sturm in his famous paper for a single homogeneous linear differential equation of the second order. If we confine our attention to an interval in which the coefficients of the differential equations are continuous real functions of the real variable x. the case here considered is more general than that of Sturm; for in order to pass from the pair of equations of the first order to a single equation of the second order we must assume not only that one of the coefficients of the equations of the first order has a continuous first derivative, but also that this coefficient does not vanish in the interval in question. The advantage of the method used in the present paper as compared with the method of Sturm consists, however, not merely in its greater generality, but also in its greater simplicity and symmetry.

In his "Grundlagen der Geometrie" (1899), Hilbert exhibited a body of axioms of three-dimensional euclidean geometry which, according to his statements, and the subsequent statements of others, involved no redundancy. Pasch, (1882) developed this geometry on the basis of the point, the linear-segment, and the planar-segment as elements, defining by means of suitable axiomatic relations, imposed upon these elements, the line and the plane. Peano (1889 and 1894) retained the point and the linear-segment as elements, but defined the line and the plane in terms of these elements. Hilbert uses as elements the point, the line, and the plane. Schur, in his paper in a recent number of the *Mathematische Annalen* (1901), points out that in view of the possibility of

a definition of a plane in terms of the other elements, Hilbert's system of axioms contains redundancy. This statement of Schur implies that the plane postulated by Hilbert is to be directly identified with the plane defined by Peano; this identification, it is proved, can in the Hilbert system I–II of axioms of connection and order be made only by use of two of the three axioms held by Schur to be redundant.

However, Hilbert's system does contain two plain redundancies. His axioms I 4: "Any three non-collinear points of a plane determine it," and II 4: "To four distinct collinear points a notation, A, B, C, D, may be given such that B lies between A and C and between A and D; and C lies between A and D and between B and D," are consequences of the remaining axioms, I, II.

The paper of Professor Moore exhibits these redundancies, and in that connection a new system of axioms of connection and order closely related to, but in various respects simpler than, those of Hilbert.

In his second paper, Dr. Hedrick makes an attempt, based on recent lectures by Professor Hilbert, to simplify the discussion of the subject. Beginning with Hilbert's new proof of Weierstrass's sufficient condition,* the author shows that Jacobi's criteria even with a slight extension can be obtained directly from the Weierstrass-Hilbert theory, by the aid of the latter's invariant integral; and Jacobi's equation itself is finally produced. Hilbert's notion of limited variation having been introduced, Legendre's necessary condition is deduced in a simple manner from the above; and the conditions known as Lagrange's, Legendre's, Jacobi's, and Weierstrass's are grouped to form necessary conditions and sufficient conditions in the cases of weak and strong minima, limited and unlimited. The discussion closes with a remark on the purpose of Hilbert's existence proof.

In his first note Dr. Eisenhart considers isotropic congruences as they arise in the study of the infinitesimal deformation of the sphere. The discussion leads to the following theorem: The middle envelope of an isotropic congruence is the adjoint of the minimal surface which is the associate surface in the infinitesimal deformation of the sphere, which is the directrix of the congruence.

* See Osgood, *Annals of Math.*, 2d Series, vol. 2, No. 3, p. 121.

Dr. Eisenhart's second note deals with the double system of imaginary lines of length zero on real surfaces. After recalling that these lines can never be the lines of curvature for the surface, that when they form a conjugate system the surface is minimal and that when they are asymptotic the surface is a sphere, he shows that the sphere and minimal surfaces are the only real surfaces whose lines of length zero have for spherical representation the rectilinear generatrices of the sphere. Further, in order that the asymptotic lines or a conjugate system on a surface may be represented upon a sphere by these generatrices, they must be lines of length zero on the surface.

Dr. Fite showed that if a group of order p^m, where p is an odd prime, contains an operator of order p^{m-2}, the group is of class k, where $k \leqq 3$; if it contains an operator of order p^{m-3}, where p is a prime greater than 3, it is of class k, where $k \leqq 4$.

The curves considered in Dr. Kasner's second paper are defined by the form to which their cartesian equation may be reduced

$$y = A_0 x^n + A_1 x^{n-1} + \cdots + A_n.$$

It is quite evident that the polar conic of any point with respect to such a parabolic curve of nth order is a parabola. The paper proves that this property is characteristic, *i. e.*, any curve whose polar conics are all parabolas can be expressed in the above form. The proof depends upon the determination of the rational integral functions $\varphi(x, y)$ which satisfy the partial differential equation

$$\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy} = 0.$$

The result is applied to a certain class of developable surfaces.

Mr. Gundersen's paper deals with different methods of measuring assemblages of points, particularly the methods of Cantor, Peano, Jordan, and Borel, and with properties of assemblages suggested by these methods. Among the results reached may be mentioned that if $F_{(1)}$ is the frontier assemblage of any linear assemblage, and $F_{(2)}$ the frontier of $F_{(1)}$ and so on, then $F_{(n)}$ is identical with $F_{(2)}$ $(n \leqq 3)$. The relation between Borel's system of measure and that of Peano and Jordan is given by the equation $s = a + \theta(A - a)$,

where s is the Borel measure, a the internal and A the external measure as defined by Peano and Jordan, and $0 \leqq \theta \leqq 1$. If the given assemblage contains its derived, the above formula takes the form $s = A$. The necessary and sufficient conditions of measurability in the different systems are discussed.

Mr. Young's paper is in abstract as follows: We denote by α-isomorphism any isomorphism of a group with itself (or with one of its subgroups), which may be obtained by putting each operator into correspondence with its αth power, and by α holomorphism any α-isomorphism which is simple. Prof. Miller has shown that α holomorphisms exist for any abelian group A (α being prime to the order of every operator in A) and further that, regarded as operators in the group of isomorphisms of A, the α-holomorphisms constitute the totality of invariant operators in the group of isomorphisms of A. It is shown for any group G that if there is a number δ such that for every pair of operators s, t of G, $s^\delta t = ts^\delta$ and $s^\delta t^\delta = (st)^\delta$, and if $\delta + 1$ is prime to the order of every operator of G, then and only then does G admit an α-holomorphism; and α may have any value of the form $x\delta + 1$ which is prime to the order of every operator of G. It is then proved that the α-holomorphisms of any group G are invariant in the group of isomorphisms of G, and certain related theorems are derived.

Professor Smith's paper treats the subject of orthogonal transformations by means of a result first obtained by Voss, that such a transformation in n variables may be resolved into n or less point-plane reflections, point and plane being in the polar relation with respect to the fundamental quadric. This theorem having been established, Cayley's formulas are derived and the generality of the types derived by Loewy proven. The method followed brings out more clearly than heretofore the completeness of the results. The question of the resolution of orthogonal transformations into involutory transformations is next discussed, the possibility of resolution into two such transformations established, and some applications given.

The object of Mr. Ford's paper is to determine an upper limit to the absolute value of the remainder after n terms of a Fourier's series representing the function $f(x)$, this upper limit to hold good for all values of x within a fixed interval (a, b) lying within the interval $(-\pi, \pi)$. The re-

sult, besides being of service in the study of the uniform convergence of Fourier's series, furnishes a means for determining an upper limit to the number of terms necessary to take in a given Fourier development in order that the error committed by stopping with the last term taken shall be less than some assigned quantity, this being true when the series is considered for *all* values of x lying within the interval (a, b) (a and b included), it being assumed throughout that $f(x)$ satisfies certain preliminary conditions (such as Dirichlet's conditions) at every point of this interval.

F. N. COLE.

COLUMBIA UNIVERSITY.

THE JANUARY MEETING OF THE CHICAGO SECTION.

THE tenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Illinois, on Thursday and Friday, January 2–3, 1902. Two sessions were held each day, opening at 10 o'clock A. M. and 2.30 o'clock P. M. In the absence of Professor E. H. Moore, President of the Society, during the morning sessions, Professor Ziwet occupied the chair on the first day and Professor White on the second day. The attendance at the various sessions was unusually large, including a number of prominent educators in the vicinity of Chicago, and the following twenty-five members of the Society:

Professor R. J. Aley, Dr. G. A. Bliss, Professor Oskar Bolza, Dr. C. L. Bouton, Professor D. F. Campbell, Mr. A. Crathorne, Professor E. W. Davis, Professor L. E. Dickson, Professor L. W. Dowling, Dr. J. W. Glover, Dr. T. P. Hall, Professor Thomas F. Holgate, Dr. H. G. Keppel, Dr. Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Miss I. M. Schottenfels, Professor G. T. Sellew, Professor J. B. Shaw, Professor E. J. Townsend, Professor C. A. Waldo, Dr. Jacob Westlund, Professor H. S. White, Professor Alexander Ziwet.

At the business session on Friday morning the following officers of the section were elected for the ensuing year: Secretary, Professor Thomas F. Holgate; Additional members of the programme committee, Professor E. J. Townsend and Professor L. W. Dowling.

The following papers were presented and read:

(1) Professor M. W. HASKELL: "A fundamental theorem in the geometry of the tetrahedron."

(2) Professor M. W. HASKELL: "A theorem for the twisted cubic analogous to Pascal's theorem."

(3) Professor M. W. HASKELL: "A special cubic transformation in space."

(4) Professor H. S. WHITE: "Note on a twisted curve connected with an involution of pairs of points in a plane."

(5) Dr. J. W. GLOVER: "On the derivation of the asymptotes of an algebraic curve from the definition" (preliminary communication).

(6) Professor ARNOLD EMCH: "Algebraic transformations of a complex variable realized by linkages."

(7) Professor L. W. DOWLING: "On the conformal representation of the isosceles triangle containing an angle of 120 degrees."

(8) Professor E. H. MOORE: "On Hilbert's plane arguesian geometry."

(9) Dr. F. R. MOULTON: "A simple non-arguesian geometry."

(10) Dr. JACOB WESTLUND: "Note on multiply perfect numbers."

(11) Dr. JACOB WESTLUND: "On the class number of a particular cyclotomic number field."

(12) Dr. C. L. BOUTON: "The equivalence of linear differential equations for a transformation of the independent variable."

(13) Dr. T. P. HALL: "An algebra of space."

(14) Professor J. B. SHAW: "Commutativity of matrices and application to the theory of linear associative algebra."

(15) Dr. H. G. KEPPEL: "A cubic three-way locus in four-fold space."

(16) Dr. J. C. FIELDS: "An equivalent of Plücker's formulæ."

(17) Professor H. B. NEWSON: "On the product of linear substitutions."

(18) Professor G. A. MILLER: "On the groups of order p^m which contain operators of order p^{m-2}."

(19) Professor L. E. DICKSON: "Some simplifications in the theory of linear groups."

In the absence of the authors, Dr. Fields's paper and Professor Haskell's third paper were read by Professor White, Professor Emch's and Professor Miller's papers by Professor Moore, Professor Newson's paper by Professor Dowling, and Professor Haskell's first and second papers by the Secretary; Professor Dickson's paper was read by title.

In addition to the papers announced on the regular programme, Professor Townsend opened a discussion on the subject of the best preparation in the undergraduate course for post-graduate study in mathematics, which developed into a discussion of the programme leading to the master's degree. The question of migration from one college or university to another was considered, and a committee consisting of Professors Waldo, Townsend, and Bolza was appointed to consider and report a scheme of equivalent requirements for candidates proceeding to their second degree with mathematics as their major subject.

An interesting feature of the meeting was a dinner served on the evening of Thursday, January 2d, in one of the University buildings, and participated in by twenty-six persons. This was followed by an exhibit by Dr. Keppel of about fifty portraits of eminent mathematicians.

Abstracts of the papers presented are as follows:

Professor Haskell's first paper contains the following theorem with some special cases: If six points are chosen at random on the six edges of a tetrahedron, one point on each edge, the four spheres determined by a vertex of the tetrahedron and the three points chosen on the edges meeting in that vertex, have one point in common. His second paper gives an analytical proof of the following theorem and its reciprocal: If eight points A, B, C, D, E, F, G, H, are chosen at random on a twisted cubic, and the eight planes $ABC, BCD, \cdots, HAB$, are considered, then the four rays of intersection of pairs of opposite planes, $ABC - EFG$, $BCD - FGH$, $CDE - GHA$, $DEF - HAB$. are generators of the same set of a ruled hyperboloid. His third paper contains a discussion of some of the interesting properties of the birational transformation which expresses the relation between a point and its harmonic conjugate with respect to a twisted cubic.

All involutions of pairs of points in a plane are divided by Bertini into four classes, distinct under the group of all Cremona transformations. The simplest of these is the relation of perspective homography. To determine this, two pairs of points are sufficient. When two pairs in such an involution are given, any third pair must satisfy two conditions. Hence if three pairs of points are given in arbitrary situation in space, a center from which they shall be projected by rays in an involution of this simplest type is subjected to two conditions, and must describe a twisted curve. In Pro-

fessor White's paper this curve is investigated, and found to be of order 5 and deficiency 0.

Dr. Glover's preliminary communication takes up the general problem of the derivation of the asymptotes of the algebraic curve $u(x, y) = 0$ by forming the equation of the tangent at the point (x, y) and then expressing the intercept and angle as fractions whose limits are investigated as the point of contact is indefinitely removed from the origin. This method leads to the usual equations for determining the values of the constants fixing the asymptotes.

In a paper appearing in 1895 in the *Comptes rendus* and also later in his book on Kinematics, Koenigs proved that the general system of (real) algebraic relations between any finite number of (real) points in space could be generated by a space linkage, that is, a linkage in which various rigid pieces are connected together at spherical pivots. Professor Emch remarks that of course this covers the case of any system of (real) algebraic relations between a finite number of (real) points which are coplanar, and further, by the usual analytical considerations, the case of any system of algebraic relations between n points in the complex plane. The single algebraic relation between two complex points may obviously be obtained by proper combination of linkages capable of effecting the determination from two arbitrary points Z_1 and Z_2, of the sum $Z_3 = Z_1 + Z_2$ on the one hand, and on the other of the product $Z_4 = Z_1 Z_2$. Professor Emch has constructed a plane linkage for the first of these two relations and makes use of a plane linkage constructed by Kleiber for the second relation. In the plane linkage the pivots are of the nature of pins perpendicular to the plane in which the linkage is operating.

Professor Dowling presented the results of a study of the integral which gives the conformal representation of the particular triangle mentioned in the title of his paper, and of the Riemann surface upon which it is an abelian integral of the first kind. The situation of the triangle corresponding to the positive half of the upper sheet is assumed and the position of the triangles corresponding to the remaining half sheets is found. The abelian integral being of deficiency two, a similar study is made of the associated independent abelian integral, and the reduction to an elliptic integral is discussed.

In his Gauss-Weber Festschrift: "Grundlagen der Geometrie," Hilbert devotes Chapter V to Desargues's theorem, and shows (page 70) that a plane geometry (I, II, III), that is, one in which the plane axioms of connection and order and the parallel axiom hold, is embodied in a spatial geometry (I, II, III) if and only if the theorem of Desargues holds in the plane geometry. In proof of the sufficiency of this condition, Hilbert develops in a plane geometry, (I, II, III, D), in which Desargues's theorem D does hold, an algebra of (directed) segments OA lying on a fixed line l, and on that line having a fixed extremity O; and on the basis of this algebra he constructs analytically the three-dimensional space (I, II, III) which contains the initial plane (I, II, III, D). In this algebra of segments, while the definitions are simple, the proofs of the correctness of some of the rules of calculation are unduly complicated. These proofs and the whole theory of the algebra Professor Moore has simplified and enriched by the introduction of a vector analysis in the plane, which embodies the segmental algebra of Hilbert as to its relations of addition, and by a suitable use of the non-essential auxiliary elements of the Hilbert definitions.

In Chapter V of Professor Hilbert's memoir on the foundations of geometry, he arrives at the following important theorem: The necessary and sufficient conditions that a plane geometry fulfilling the projective axioms of plane geometry may be a part of (or set in) a spatial geometry of more than two dimensions fulfilling the projective axioms, is that Desargues's theorem shall be fulfilled. In order to prove that the fulfillment of Desargues's theorem is necessary, Hilbert remarked the well known fact that the theorem can be proved from a spatial geometry of more than two dimensions fulfilling the axioms in question. He then exhibited a non-arguesian geometry fulfilling the plane axioms in question. His example is somewhat complicated, involving the intersections of an ellipse and a system of circles. In short, it depends upon the solution of simultaneous quadratic equations. Dr. Moulton's paper proves this part of the theorem by the exhibition of an extremely simple geometry fulfilling the axioms in question, and in which Desargues's theorem is not true. The demonstrations involve only linear equations. The geometry consists of ordinary euclidean points in a plane, but the lines are described in terms of the euclidean lines as follows:

The lines

$$x - a = \delta_{y,\theta} \tan \theta y,$$

where $\delta_{y,\theta} = 1$ if $y < 0$ or if $y > 0$ and $\theta < 0$, and $\delta_{y,\theta} = 2$ if $y > 0$, and $\theta > 0$, are to be considered as the straight lines in the new geometry. It is easily seen that they fulfill the projective plane axioms, and that in this geometry Desargues's theorem is not true.

In a paper presented at the meeting of the Chicago section in April last, and published in the *Annals of Mathematics*, series 2, volume 2, Dr. Westlund determined all multiple perfect numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2^{a_2} p_3$. In the present paper all numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2 p_3 p_4$, where p_1, p_2, p_3, p_4 are distinct primes, are determined.

In Dr. Westlund's second paper it is shown how the first factor of the class number of the cyclotomic number-field $k(e^{2\pi i/p^x})$ can be determined. The results are then applied to the particular cases $x = 2, 3$.

Two linear differential equations are said to be equivalent for change of the independent variable when it is possible to transform the first into the second by a change of the independent variable alone. Dr. Bouton's paper gives the necessary and sufficient conditions for this equivalence in invariant form in the case of equations of the nth order. The condition for the general equation of the second order $y'' + 2p_1(x)y' + p_2(x)y = 0$, is found as follows: Compute the invariants

$$I_1 = \frac{(p_2' + 4p_1p_2)^3}{p_2^3}, \quad I_2 = \frac{p_2'' + 4p_1'p_2 + 10p_1p_2' + 24p_1^2p_2}{p_2^2}.$$

For a given equation these are in general functions of x, so that x may be eliminated, giving $I_2 \equiv \psi(I_1)$. The same thing can be done for the second given differential equation. The neces-ary and sufficient condition for the equivalence of the two differential equations is that the two functions ψ so determined shall be the same. Various special cases require separate consideration. For the equation of the nth order there are $n - 1$ such functions ψ.

In Dr. Hall's paper, a vector algebra is sought in which each algebraic expression represents a geometric configuration, and each algebraic operation a motion in space.

From the laws of motion the laws of algebraic operations are derived, and the resulting algebra is found to include quaternions as a special form. In two-fold space it leads to the plane complex number and in all hyperspace it has a quaternion-like rotor.

It is shown in Professor Shaw's paper that, by a very simple form of the square representation of a matrix, the condition that φ_j shall satisfy the equation

$$\varphi_j\varphi_i = \varphi_i\varphi_j$$

can be easily found graphically. The totality of matrices on the same ground commutative with φ_i is determined, and also the relations of the various independent components of the most general commutative matrix. The two classes of components are considered.

In the application, the use of the preceding in determining linear associative algebras is developed, and the full structure of any such algebra exhibited. The matrix explanation of the Lie-Scheffers theory of multiple algebras is brought out, and the real basis of the classification into quaternion and non-quaternion forms is shown. The place of Peirce's study is also shown. Certain developments of the theory in extension of it are also presented.

In Dr. Keppel's paper the cubic three-spread containing a double plane in four-fold space is considered. This locus has a single infinity of plane generators, these generators meeting each other in the points of the double plane while consecutive plane generators intersect in the points of a conic in this double plane. A section of this locus by a three-flat is a ruled cubic surface, the section of the double plane being the double line. The ruled cubic surface will have two real distinct, two imaginary, or two coincident pinch points according as this line meets the conic in two real distinct, in two imaginary, or in two coincident points. Seven different ways of generating the locus geometrically are found, and sections of these by a three-flat give all the well known different ways of generating the ruled cubic surface in three-fold space. One way of generating the locus, namely, by means of two plane point fields collinearly related, gives the locus as the envelope of its double infinity of tangent three-flats, and this method provides a simple representation of the locus on the plane. The reciprocal of the locus is the general cubic three-spread in four-fold space,

and systems of these cubic surfaces lie on the cubic three-spread here considered.

Dr. Fields's paper may be characterized as follows: If $F(z, u) = 0$ be the equation to an algebraic curve of degree n in u, the n branches corresponding to a value $z = a$ will group themselves in a certain number r_κ of cycles of orders $v_1^{(\kappa)}, \cdots, v_{r_\kappa}^{(\kappa)}$. The lowest exponents in the developments of the function $F_u'(z, u)$ corresponding to these cycles we shall indicate by $m_1^{(\kappa)}, \cdots, m_{r_\kappa}^{(\kappa)}$. In like manner, if $G(\xi, \eta) = 0$ be the equation of our curve transformed to tangential coördinates, we shall have a number $\bar{r}_\kappa$ of cycles corresponding to a value $\xi = a_\kappa$ of orders $\bar{v}_1^{(\kappa)}, \cdots, \bar{v}_{\bar{r}_\kappa}^{(\kappa)}$, respectively, and for the function $G_\eta'(\xi, \eta)$ a set of lowest exponents $\bar{m}_1^{(\kappa)}, \cdots, \bar{m}_{\bar{r}_\kappa}^{(\kappa)}$. Plücker's formulæ may then be replaced by the three formulæ

$$\sum_\kappa \sum_{s=1}^{r_\kappa} m_s^{(\kappa)} v_s^{(\kappa)} = 0, \quad \sum_\kappa \sum_{s=1}^{\bar{r}_\kappa} \bar{m}_s^{(\kappa)} \bar{v}_s^{(\kappa)} = 0,$$

$$\sum_\kappa \sum_{s=1}^{r_\kappa} (v_s^{(\kappa)} - 1) - \sum_\kappa \sum_{s=1}^{\bar{r}_\kappa} (\bar{v}_s^{(\kappa)} - 1) = 2(n - \bar{n}),$$

where the respective summations are extended to all values of the variable z, or to all values of ξ as the case may be, the values $z = \infty$ and $\xi = \infty$ included. The first two formulæ follow from the theorem that a rational function has as many zeros as infinities, the last formula from the equality of the genus of the equation $F(z, u) = 0$ and that of the transformed equation $G(\xi, \eta) = 0$.

The three formulæ in the above form appear of themselves in a general algebraic theory of the algebraic functions which the author has developed and which he expects to publish shortly. They present themselves in the theory of the algebraic curves possessing higher singularities under those circumstances where in the ordinary theory of algebraic curves Plücker's formulæ occur, and furthermore in such a way that the customary question in regard to the combination of simple singularities to which a higher singularity is equivalent becomes superfluous, at least from an algebraic standpoint.

In Professor Newson's paper it is shown that the value x_i in the product of two linear substitutions T and T_1 in n

variables is proportional to the determinant formed by bordering the determinant T^{-1}, the inverse of T, vertically by the variables of T, and horizontally by the coefficients of the ith equation in T_1. The paper will be published in an early number of the *Annals of Mathematics*.

Burnside determined the groups of order p^m, p being any prime number, which contain a cyclic invariant subgroup of order p^{m-2}. In a recent article published in the *Transactions* of this Society, Professor Miller determined those groups of order p^m which contain the abelian group of type $(m-2, 1)$. His present paper is devoted to a study of the remaining groups of order p^m which contain operators of order p^{m-2}. It is proved that there are just two such groups when $p > 2$ and $m > 5$, while there are five when $p = 2$ and $m > 5$. In the first case $(p > 2, m > 5)$ there are therefore 15 groups which contain operators of order p^{m-2}—four are abelian, five others contain an invariant cyclic subgroup of order p^{m-2}, while the remaining six do not have either of these properties. Each of these 15 groups is conformal with some abelian group. In the second case $(p = 2, m > 5)$ there are 33 groups which contain operators of order 2^{m-2}—four of these are abelian, eighteen others contain an invariant cyclic subgroup of order 2^{m-2}, while eleven do not possess either of these properties. These eleven are composed of G_8, G_9, G_{12}, G_{13}, G_{14}, and G_{17} of the article mentioned and the five groups of the present article. The full paper will be offered for publication to the *Transactions*.

In recent papers in the *Transactions* of this Society and in the *Proceedings of the London Mathematical Society*, Professor Dickson investigated several classes of linear groups in an arbitrary realm of rationality. The earlier methods applicable to finite fields were replaced by developments involving only the rational processes of algebra. Among the groups considered are those defined by quadratic invariants. The latter, for finite fields of modulus $p > 2$, may be given the canonical forms

$$\xi_1^2 + \cdots + \xi_{m-1}^2 + \xi_m^2 \qquad (m \text{ even or odd}),$$

$$\xi_1^2 + \cdots + \xi_{m-1}^2 + \nu\xi_m^2 \qquad (m \text{ even}),$$

where ν is a not-square. As another set of canonical forms, for finite fields, we may take

$$f \equiv \xi_1\eta_1 + \cdots + \xi_m\eta_m, \qquad \xi_0^2 + f, \qquad \xi_0^2 - \nu\eta_0^2 + f.$$

The conditions for the invariance of f under a linear transformation are bilinear in the coefficients and therefore may be treated by rational processes. Similar remarks hold for a treatment of the remaining two forms by induction from $m-1$ to m. The second set of canonical forms defines groups in an arbitrary realm whose structures were determined in the papers cited. Moreover, the second set proves more advantageous than the first set in the question of the representation of the linear groups as transitive substitution groups. The first set was employed in the *American Journal of Mathematics*, October, 1901. For the second set, the investigation becomes simpler and the results appear in a simpler and more natural classification.

Not only for these groups, but for other classes, it appears that the simplest methods are those applicable to an arbitrary realm, viz., methods which depend essentially upon rational operations only.

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Secretary of the Section.

VECTOR ANALYSIS.

Vector Analysis. A text-book for the use of students of mathematics and physics, founded upon the lectures of J. WILLARD GIBBS, Ph.D., LL.D., Professor of Mathematical Physics in Yale University. By EDWIN BIDWELL WILSON, Ph.D., Instructor in Mathematics in Yale University. (Yale Bicentennial Publication.) New York, Charles Scribner's Sons, 1901. 8vo. xx + 436 pp.

IT is well known that Professor Gibbs's "Elements of vector analysis," a pamphlet of 83 pages, printed in 1881–84 for the use of his students, although not published for general circulation, attracted somewhat wide attention. Thus, in particular, Mr. Oliver Heaviside adopted Professor Gibbs's system with but slight modifications and expounded it very fully in his "Electromagnetic Theory" (1893); this again formed the basis, to a large extent, of Professor Föppl's "Einführung in die Maxwell'sche Theorie der Elektricität, mit einem einleitenden Abschnitte über das Rechnen mit Vectorgrössen in der Physik" (1894). But vector analysis, as conceived by Professor Gibbs, is not merely a method

for studying electricity and magnetism; it finds its application in all branches of mathematical physics and has, besides, a strong claim to the attention of the pure mathematician. Dr. Wilson's volume, which is based on a course of lectures delivered by Professor Gibbs in 1899–1900, is therefore exceedingly welcome, not only as the first generally accessible authentic record of Professor Gibbs's admirable system, but also as an easy and simple introduction to vector analysis, sufficiently complete to satisfy the needs of applied science and sufficiently exact and rigorous to satisfy the student of pure mathematics.

The additions to the theory found in this volume as compared with the pamphlet of 1881–84 are perhaps not very extensive, though some are quite important. The student should not be deterred by the bulkiness of Dr. Wilson's book, as this is due partly to the use of lavishly open print and partly to the author's effort, specially noticeable in the first half, of making the subject easily intelligible by supplying numerous illustrations and applications.

In his address on "Multiple Algebra," before the section of mathematics and astronomy of the American Association for the advancement of science in 1886 (Proceedings, Vol. 30, pp. 37–66), Professor Gibbs has set forth briefly the relation of his system to Grassmann's Ausdehnungslehre, Hamilton's quaternions, Cayley and Sylvester's theory of matrices, and Benjamin Peirce's linear associative algebra. This paper explains and illustrates very happily the simplicity and naturalness of his method as far as fundamental principles are concerned. But the value of vector methods is still so little appreciated that it may not be superfluous to call attention once more to their importance by briefly reciting some of the principal features of the system presented in Dr. Wilson's book.

Mathematical physics, in its broader aspect, has to deal with *fields* of *point functions*. By somewhat generalizing Lamé's definition it can be said that a point function is any quantity, or aggregate of quantities, whose value depends on the position of a point.* The simplest point function is a *scalar* whose value at any point of the field is given by a single (real) number. Next in importance comes the *vector*,

* Professor Gibbs uses for "point function" the expression "function of position in space" which is somewhat unwieldy. The present writer would also suggest that Dr. Wilson might have done well to speak of a point function, that is a vector quantity, simply as *vector*, and not as *vector function*, so as to reserve the latter term for a function of a vector, as in the expression linear vector function.

a point function having not merely numerical magnitude but also direction (and sense); its value at any point is represented geometrically by a rectilinear segment and given analytically by its components, *i. e.*, three scalars, each associated with a fixed direction. As examples of point functions of a higher order occurring in physics it will suffice to mention *pure strain*, with its six components (or three principal extensions in definite directions), and *homogeneous strain*, depending upon nine scalars.

The theory of scalar fields is nothing but an immediate generalization of the theory of the potential, a generalization on which Professor Gibbs very properly lays particular stress in his treatment of the potential (pp. 205–248) which forms one of the most interesting parts of Dr. Wilson's work.

Every scalar field gives rise to a vector field, *viz.*, the field of the vector that represents the *gradient* or *space derivative* ∇V of the scalar V. Thus, in the theory of the ordinary potential, the space derivative of the potential is the corresponding force. But the vector that results in this way from a scalar is a particular kind of vector: a vector whose *curl* is zero; the field of such a vector, and the vector itself, is said to be *potential*, or *irrotational*. It is therefore necessary to study vectors by themselves.

The idea of vectors and their geometrical addition is now familiar to everybody from the elements of mechanics. Dr. Wilson follows Mr. Heaviside in using heavy-faced (Clarendon) type to denote vectors. This departure from Professor Gibbs's original notation by small Greek letters is commendable as it leaves the Greek alphabet free for other uses. A vector $\mathbf{A}$ whose components along three mutually rectangular axes are A_1, A_2, A_3 is then expressed in the form

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the axes. This means that a vector is an extensive magnitude depending on three independent units.

The *scalar product* of two vectors $\mathbf{A}$, $\mathbf{B}$ is defined as the product of their lengths A, B into the cosine of the included angle θ:

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta;$$

this *dot* product is a scalar, *i. e.*, an ordinary algebraical quantity (real number). The *vector product* is similarly defined as

$$\mathbf{A}\times\mathbf{B} = AB\sin\theta;$$

this *cross* product, being the area of the parallelogram of the two vectors, can be represented by a vector at right angles to the plane of the two vectors. Both of these quantities are familiar from their frequent occurrence in mechanics, the scalar product as the work of a force, the vector product as the moment of a couple. Both products obey the ordinary rules of algebra except that the vector product is not commutative ($\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$) and that the vanishing of either product does not necessarily mean that one of the factors vanishes. For the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we have evidently :

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \quad \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{j} = 0,$$

and

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0, \quad \mathbf{j}\times\mathbf{k} = -\mathbf{k}\times\mathbf{j} = \mathbf{i},$$
$$\mathbf{k}\times\mathbf{i} = -\mathbf{i}\times\mathbf{k} = \mathbf{j}, \quad \mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i} = \mathbf{k}.$$

It follows that if $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$ we have

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3,$$

$$\mathbf{A}\times\mathbf{B} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k}.$$

The development of these simple ideas constitutes the first two chapters (pp. 1–114) of Dr. Wilson's book ; they correspond to the first thirteen pages of Professor Gibbs's pamphlet. The algebra so obtained, especially when combined with the equally simple differentiation of a vector with respect to a scalar variable as given in the first twenty pages of the next chapter, suffices for the treatment of a large part of mechanics and admits of interesting applications to geometry.

Chapters III and IV (pp. 115–259), corresponding to pp. 14–39 of the pamphlet, are devoted to the differential and integral calculus of vectors. The central idea in these chapters is Hamilton's operator ∇ for which the pronunciation *del* is here proposed instead of *nabla* (p. 138). This operator is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z},$$

so that, applied to a scalar, $V(x, y, z)$, it produces the vector that represents the *slope* or *gradient* of the point function V. The direction of ∇V is normal to the level-surface

$V(x, y, z) = \text{const.}$; its magnitude is the rate at which V changes as we pass from one level surface to the next along the normal. This vector ∇V, being independent of the coördinate axes, can be regarded as the *space derivative* of the scalar V. It is easy to see that, if $\mathbf{r}$ be the radius vector drawn from a fixed origin to the point (x, y, z), we have

$$d\mathbf{r} \cdot \nabla V = dV.$$

The space derivative of a vector $\mathbf{V}$ can be defined in precisely the same way. But as its interpretation involves the idea of a new kind of product, its consideration is reserved for the latter part of the book (p. 404). In the place of this space derivative $\nabla\mathbf{V}$, two other quantities, of the greatest importance in mathematical physics, are introduced, the *divergence* and the *curl* of the vector $\mathbf{V}$. These are most readily obtained by regarding the operator ∇ as an actual vector and forming its scalar and vector products with the vector $\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$, viz.:

$$\begin{aligned}
\nabla \cdot \mathbf{V} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\
&= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = \operatorname{div} \mathbf{V}, \\
\nabla \times \mathbf{V} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\
&= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k} \\
&= \operatorname{curl} \mathbf{V}.
\end{aligned}$$

The meaning of these quantities is well explained by Dr. Wilson; the formulæ obtained by applying the operators ∇, $\nabla\cdot$, $\nabla\times$ to the sum or product of two or more scalars or vectors and by repeating these operations are fully developed.

In the integral calculus of vectors we are led naturally and with remarkable simplicity to line, surface, and volume integrals and to the theorems of Gauss, Green, and Stokes which appear here in a form which interprets itself and expresses most simply and directly their actual meaning.

Among the applications here discussed Professor Gibbs's original treatment of so-well worn a subject as the potential

is specially noteworthy. According to the ordinary definition, the integral

$$\int \frac{V}{r}\, d\tau = \iiint \frac{V(x, y, z)\, dx\, dy\, dz}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}}$$

represents the potential at the point (x_1, y_1, z_1) of a mass distributed with density V throughout the region τ; it is a scalar function of the variables x_1, y_1, z_1. An integral of this form, whatever may be the meaning of the scalar function V, is called by Professor Gibbs the *potential* of V; in symbols:

$$\text{Pot } V = \int \frac{V}{r}\, d\tau .$$

It can be proved, by means of Gauss's method of giving to the region τ a slight displacement in each of the coördinate directions, that

$$\nabla \text{ Pot } V = \text{Pot } \nabla V.$$

This point function which in the case when V is the density of a mass represents the Newtonian attraction due to this mass at the point (x_1, y_1, z_1) is called the *Newtonian* of V and denoted by the symbol New V:

$$\text{New } V = \nabla \text{ Pot } V = \int \frac{\mathbf{r} V}{r^3}\, d\tau.$$

If in the integral defining Pot V we replace the scalar V by a vector $\mathbf{V}$ we obtain a vector

$$\text{Pot } \mathbf{V} = \int \frac{\mathbf{V}}{r}\, d\tau.$$

Operating on this vector by means of $\nabla \times$ and $\nabla \cdot$, two more functions are obtained called the *Laplacian* and *Maxwellian* of $\mathbf{V}$ and denoted as Lap $\mathbf{V}$ and Max $\mathbf{V}$. These functions, as well as some of those arising from their combination, play a most important part in mathematical physics.

It should be noticed that this whole theory rests on but three simple fundamental operations: the scalar and vector products of two vectors and the space derivative of a scalar. With characteristic skill Professor Gibbs has selected, out of the broad field of multiple algebra and space analysis, just those operations which are most useful on account of their application in physics.

Dr. Wilson's presentation of these subjects is on the whole very clear and sufficiently exact. The idea of scalar and vector fields and their varieties might perhaps have been made more prominent, somewhat as Professor Föppl does in his "Geometrie der Wirbelfelder" (1897). As the book is distinctly designated as a text-book, it would have been better to make shorter chapters and sections, introduce more section headings and distinguish, externally, between what is essential and fundamental and what is mere illustration and application. The very large and heavy integral signs used here as in so many other books are neither handsome nor always necessary. Nor is there any necessity for three integral signs in a volume integral in which the volume element dv is not broken up into its linear factors; the three signs become desirable only when dv is replaced by $dx\,dy\,dz$ or $r^2 \sin\theta\, dr\, d\theta\, d\varphi$. A similar remark applies to surface integrals.

The second half of Dr. Wilson's book (Chapter V: linear vector functions, pp. 260–331; Chapter VI: rotations and strains, pp. 332–371; Chapter VII: miscellaneous applications, pp. 372–436) is almost entirely devoted to the linear vector function and its applications. The beginner will probably find it more difficult reading than the first half; the originality of the treatment will make it so much more interesting to the more advanced student.

It is here impossible to do more than indicate the fundamental idea underlying Professor Gibbs's treatment of the linear vector function.

If a vector be given in the usual form $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, any other vector $\mathbf{r}'$ can be written in the form

$$\begin{aligned}\mathbf{r}' &= c_1 x\,\mathbf{i} + c_2 y\,\mathbf{j} + c_3 z\,\mathbf{k} \\ &= c_1 \mathbf{i}\mathbf{i}\cdot\mathbf{r} + c_2 \mathbf{j}\mathbf{j}\cdot\mathbf{r} + c_3 \mathbf{k}\mathbf{k}\cdot\mathbf{r},\end{aligned}$$

since $x = \mathbf{i}\cdot\mathbf{r}$, $y = \mathbf{j}\cdot\mathbf{r}$, $z = \mathbf{k}\cdot\mathbf{r}$. For the sake of brevity this may be written

$$\mathbf{r}' = (c_1\mathbf{i}\mathbf{i} + c_2\mathbf{j}\mathbf{j} + c_3\mathbf{k}\mathbf{k})\cdot\mathbf{r},$$

the expression in parenthesis being regarded as an operator that transforms $\mathbf{r}$ into $\mathbf{r}'$. More generally, any expression of the form

$$\Phi = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots,$$

in which $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{b}_1, \mathbf{b}_2, \cdots$ are vectors, can be regarded as an

operator transforming a vector into another vector; for the expression

$$\Phi\cdot\mathbf{r} = (\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots)\cdot\mathbf{r} = \mathbf{a}_1\mathbf{b}_1\cdot\mathbf{r} + \mathbf{a}_2\mathbf{b}_2\cdot\mathbf{r} + \cdots$$

always represents a definite vector since $\mathbf{b}_1\cdot\mathbf{r}$, $\mathbf{b}_2\cdot\mathbf{r}$, $\cdots$ are mere scalars. The nature of the *indeterminate products* $\mathbf{a}_1\mathbf{b}_1$, $\mathbf{a}_2\mathbf{b}_2$, $\cdots$ or *dyads*, as Professor Gibbs calls them, results from the convention that any two such products are to be equal if, and only if, acting as operators on the same vector $\mathbf{r}$, they produce the same vector $\mathbf{r}'$.

The sum of any number of dyads,

$$\Phi = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots,$$

is called a *dyadic*. Thus a dyad as well as a dyadic has no direct geometrical meaning; it is a mere operator whose close relation to quaternions and matrices is obvious.

The linear vector function being defined by the property that $f(\mathbf{r}_1 + \mathbf{r}_2) = f(\mathbf{r}_1) + f(\mathbf{r}_2)$, from which the property $f(m\mathbf{r}) = mf(\mathbf{r})$ readily follows for any scalar m if f be assumed continuous, it can be shown that a linear vector function can always be represented by a dyadic. By expressing the vectors forming the dyadic in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the dyadic can be reduced to the *nonion* form:

$$\begin{aligned}\Phi = {} & a_{11}\mathbf{ii} + a_{12}\mathbf{ij} + a_{13}\mathbf{ik} \\ & + a_{21}\mathbf{ji} + a_{22}\mathbf{jj} + a_{23}\mathbf{jk} \\ & + a_{31}\mathbf{ki} + a_{32}\mathbf{kj} + a_{33}\mathbf{kk},\end{aligned}$$

or more generally to the form

$$\begin{aligned}\Phi = {} & b_{11}\mathbf{aA} + b_{12}\mathbf{aB} + b_{13}\mathbf{aC} \\ & + b_{21}\mathbf{bA} + b_{22}\mathbf{bB} + b_{23}\mathbf{bC} \\ & + b_{31}\mathbf{cA} + b_{32}\mathbf{cB} + b_{33}\mathbf{cC},\end{aligned}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as well as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are any three non-coplanar vectors. Putting

$$b_{11}\mathbf{A} + b_{12}\mathbf{B} + b_{13}\mathbf{C} = \mathbf{l},\quad b_{21}\mathbf{A} + b_{22}\mathbf{B} + b_{23}\mathbf{C} = \mathbf{m},$$
$$b_{31}\mathbf{A} + b_{32}\mathbf{B} + b_{33}\mathbf{C} = \mathbf{n},$$

it appears that any dyadic can be written as a sum of (at most) three terms:

$$\Phi = \mathbf{al} + \mathbf{bm} + \mathbf{cn},$$

where either $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ or $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ can be selected arbitrarily (but not coplanar). If after choosing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ arbitrarily, the vectors $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ result non-coplanar, the dyadic is *complete;* such a dyadic can produce *any* vector $\mathbf{r}'$ by operating on a properly

chosen vector $\mathbf{r}$. If $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ result coplanar, the dyadic can be reduced to the sum of two dyads and is called *planar;* it transforms any vector $\mathbf{r}$ into a vector lying in the plane of the first factors of the two dyads. If $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ result collinear, the dyadic can be reduced to a single dyad and is called *linear;* it transforms every vector into a vector collinear with the first factor of the single dyad.

The discussion on pp. 271–275 of the nature of the indeterminate product $\mathbf{ab}$ and its relation to the scalar product $\mathbf{a} \cdot \mathbf{b}$ and the vector product $\mathbf{a} \times \mathbf{b}$ is interesting theoretically. For all further developments the reader must be referred to the book itself. It must here suffice to mention the introduction of the "double products" of dyads and dyadics (pp. 306–315), which are not to be found in the pamphlet of 1881–84.

The applications treated in Chapters VI and VII give ample opportunity for practice in the theory developed in the earlier part of the work; they relate to rotations and strains, quadric surfaces, the propagation of light in crystals, and the curvature of surfaces. The brief section on light waves in non-isotropic bodies will arouse in every reader the wish that Profe-sor Gibbs may soon find it possible to publish his lectures on the electro-magnetic theory of light.

The last section, on the curvature of surfaces, is preceded by a short discussion of variable dyadics; and here we find the interpretation of the space derivative of a vector, $\nabla\mathbf{V}$, as a dyadic.

In a work departing from familiar mathematical methods as much as does the present it is but natural that a large number of new terms and notations are introduced. These have been selected with unusual skill and good sense; indeed, this is one of the strong points of Profes-or Gibbs's system. Perhaps the expressions degree of nullity (of a dyadic) and idemfactor, a remnant of Sylvester's always ingenious, but somewhat extravagant, terminology. make an exception. The former of these terms seems hardly necessary, the latter might perhaps be replaced by unit dyadic.

It is very much to be hoped that Dr. Wilson's work will become a widely used text-book, especially as a general introduction to mathematical physics. Its usefulness for this purpose would have been much enhanced by a good index, by better figures, and by the use of all those external aids now so common in carefully printed mathematical text-books.

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EUCLID'S ELEMENTS.

Euclid und die sechs planimetrischen Bücher. Mit Benutzung der Textausgabe von Heiberg. Von Dr. Max Simon. Leipzig, Teubner, 1901. 8vo. vi + 141 pp.

The Contents of the Fifth and Sixth Books of Euclid. Arranged and explained by M. J. M. Hill, F.R.S. Cambridge University Press, 1900. 4to. xix + 243 pp.

It is probable that, whether we are glad of it or not, Euclid has been banished from our American schools, never to return. Yet there is a real need for better professional knowledge among our teachers of geometry, and for this reason we welcome Dr. Simon's book, which fairly bristles with remarks helpful even to those teachers who rely upon the most iconoclastic of our text-books.

The book begins with a short account of Euclid's writings, and an elaborate bibliography of the "Elements." The choice of books to which reference is made, seems a little arbitrary. For instance, the author mentions (page 14) the fifteen editions of Legendre, published between 1794 and 1852, under the head of "Efforts to supersede Euclid," but passes over in silence such important work as the "Syllabus" of the "Association for the Improvement of Geometrical Teaching," not to speak of any more recent literature. In fact, he seems to have cared little for English sources, except Simson; for, though he mentions Hoüel's defence of Euclid against Legendre, he has no word for that most truly charming bit of textbook criticism, Dodgson's "Euclid and his modern rivals."

The introduction is followed by the twenty-two definitions of book I, with a note to each. In referring to the complaint frequently urged that some of Euclid's definitions would be no help to any one who had previously no idea of the objects in question, the author quotes from Lambert to the effect (page 25) that Euclid was doubtless aware of this, and in giving the definitions he was merely acting as the artisan who shows his apprentice around the shop, telling him the names of the various implements. The remarks upon the definitions are for the most part good, especially those which deal with the "point." In defining the angle, Dr. Simon uses the word "Biegung" instead of the more common "Neigung," and defends the innovation at some length. We are not convinced, however, that this is a change for the better, for an idea of cur-

vature or bending seems to lurk about "Biegung" which is certainly not wanted in the definition of a plane angle. The notes upon the axioms are less satisfactory. The author leaves us in doubt about his own opinion as to the possibility of proving that all right angles are equal, merely showing that the proof of Geminos is inadequate.

The criticisms of the propositions in the first book are perhaps the most instructive of any. For instance, he points out how the proof of the first congruence theorem for triangles depends upon the axiom of free mobility, or the homaloidal nature of space. It is a pity that he does not dwell at even greater length upon this point, for Hilbert* and Veronese† assume this theorem as an axiom. Yet something may still be said upon the side of those who look upon geometrical congruence as meaningless, except by means of possible superposition, and hence regard the introduction of such an axiom as superfluous. ‡

On page 51 is the proof of the proposition that an exterior angle of a triangle is greater than either of the opposite interior angles. The author remarks that this was used by Legendre to show that the sum of the angles of a triangle cannot be greater than two right angles; but many years before, Saccheri turned it to a similar use. Dr. Simon neglects to show how this proof may break down when a straight line is closed.

The notes on books two and three are less copious, with the interesting exception of pages 87–90, where we have an account of the dispute over the magnitude of the angle between two tangent circles. At the beginning of book five are some particularly instructive notes upon the intimate connection between the euclidean theory of proportion, and our modern idea of irrational numbers. The author goes so far as to say (page 109), "It appears, in my opinion, indisputably, from the fifth book that they (the Greeks) possessed the idea of number with complete sharpness, nearly identical with Weierstrass's conception of it." On the next page occurs a singular omission, where the author totally fails to grasp the significance of Archimedes's axiom. He merely calls it a postulate to the effect that lengths may be compared as to magnitude, and used in arithmetical processes. But the axiom of Archimedes is primarily one of continuity, and what is more it is possible to construct non-

* Hilbert, Die Grundlagen der Geometrie, p. 12.

† Veronese, Grundzüge der Geometrie von mehreren Dimensionen, p. 260.

‡ Cf. Russell, The Foundations of Geometry, pp. 147–158.

archimedean number systems, and geometries.* True, it is hard to imagine just what such geometries are like. We are in the habit of assuming † that if a variable increase unceasingly it will either approach a limit or eventually surpass any chosen value; but if in non-archimedean geometry we lay off the same length an endless number of times on a straight line, neither of these will occur, the latter being prohibited by definition, while the former is at variance with our definitions of fixed length and limit.

One more omission is worth speaking of: it occurs in the note to the eighteenth proposition in book six. This is a theorem to show that upon any segment of a straight line, we may construct a figure similar to a given figure, and similarly placed. The construction consists, of course, of a composition of similar triangles; but a rigorous proof should be explicitly based on the method of mathematical induction, which is tacitly used, and also show that the construction is unique. Euclid naturally goes into neither of these subtleties, but their absence should certainly be noticed in a careful commentary.

In conclusion, it is worth mentioning that Dr. Simon's commentaries are all of a historical, philological, or mathematical nature, and untrammeled by pearls of pedagogical wisdom. This is a wise limitation, for an equally adequate treatment of such debatable matter as methods of teaching Euclid would have changed the book from a compact to a voluminous one, besides being foreign to the purpose of the historical series of which it forms a part.‡ Yet it is a very human work withal, as may be seen from the following anecdote, which the writer narrates (page 19). "I found a construction for a tangent to the 'limiting circle' which seemed to me equally simple and new for our old circle. I announced it in Munich where were assembled one hundred mathematicians, nearly all teachers of distinction in the universities; and neither I nor any one else had a suspicion that it was Euclid's construction." (Book 3, proposition 17.)

The fact, mentioned above, that Euclid no longer presents a living issue in American education makes it difficult to pass a fair judgment on Dr. Hill's book. We are inclined to look upon attempts to "temper the wind to the shorn

* Hilbert, l. c. cit., p. 25; Dehn, "Die Legendre'schen Sätze über die Winkelsumme im Dreieck," *Math. Annalen*, vol. 53.

† Conf. Osgood, Introduction to Infinite Series, p. 14.

‡ Abhandlungen zur Geschichte der mathematischen Wissenschaften.

lamb" by elucidating particular spots in the "Elements" as a waste of time and energy. But we have no right to start from such an assumption in considering the present work. The simple fact is that Euclid has been, and will long continue to be, the foundation of geometrical knowledge in that nation which has produced Newton and Cayley and Sylvester. Parts of Euclid are undoubtedly too difficult for beginners, and the book before us attempts to remove the greatest of these difficulties, the theory of proportion. In American books we seek to reach this end by an appeal to the analogy of algebra, but herein we depart entirely from Euclid's pattern. Professor Hill adopts a different, and more euclidean device: the relative multiple scale.

Let A and B be two magnitudes which are supposed to be capable of comparison as well as their multiples. The order of succession of the various multiples of A and B is called their relative multiple scale, and may be represented symbolically by $rA \lessgtr sB$ where r and s are positive integers. The relative multiple scales of A, B and C, D are equal if, for every positive integral value of r and s, $rA \lessgtr sB$ is a sufficient condition for $rC \lessgtr sD$. A simple geometrical diagram is introduced to exhibit the three possible relations and this diagram is continually presented to give the learner a clear idea of what he is doing. Two sets of magnitudes having equal multiple scales are said to be proportional, and the transformations which leave the proportional relation unaltered constitute a large part of the book. With perfect propriety, the author lays great stress upon the axiom of Archimedes, and there is throughout a commendable consistency in the method and spirit of the work.

Such is the general scheme and, granted the author's aim, and point of view, it is about the best that could be adopted. In some of the details, the book is almost incredibly careless.

The first proposition (page 2) is a proof of the commutative and associative laws in the multiplication of positive integers, but the proofs are based upon the assumption that these laws hold in the case of addition. On page 6 is the proof that if a and b are positive integers, while $a > b$, and R is a magnitude $aR > bR$. It is tacitly assumed that $bR + cR > bR$, or that R obeys axiom 9, "The whole is greater than its part."

A multiple scale is defined (page 12) as a set of magnitudes, while a relative multiple scale is a portion of a diagram (page 14). On page 29 is the corollary "If ABC be a

triangle, and if the sides AB, AC be cut by any straight line parallel to BC, then the sides AB, AC are divided proportionally." This is given before the word "proportionally" is defined.

Section 3, pages 33–36, is entitled "A Chapter on Ratio," and from it we glean the following facts: The reader is supposed to have an innate idea of relative magnitude. "The ratio of one magnitude to another (which must be of the same kind as the first) is the relative magnitude of the first compared with the second." Having now a clear and precise idea of what a ratio is, from this soul-satisfying definition, we are prepared for the next step, which is this: If we accept the axiom that a number ρ may always be found of such a sort that the relative multiple scale of ρ and 1 is equal to that of the two comparable magnitudes A and B, then ρ may be taken as the measure of the ratio of any pair of magnitudes having the same scale as A and B. It is interesting to notice that the author himself seems to have a certain diffidence about these definitions, for he does not use them in any of his subsequent work, but sticks to the safe path of the relative multiple scale. Apparently their only use in the whole chapter is that it is convenient to use the words "ratio" and "proportion" while Euclid's definition of the latter is cumbersome.

As a last error in detail, we notice the following definition on page 73:

"If four straight lines (presumably in the same plane) be cut by any transverse in four harmonic points, they are called four harmonic lines, or said to form a harmonic pencil." This may be interpreted in two ways. If the author means that they either form four harmonic lines or a harmonic pencil, the former seems a perfectly useless definition of four harmonic tangents to a conic; if, on the other hand, he looks upon the two forms of words as synonymous, and this seems the more natural interpretation, he is departing from the usual English practice of defining a pencil as composed of coplanar and concurrent lines.*

To sum up: the book seems to us well planned but carelessly executed. It has, of course, no message for the average American teacher, and whoever uses it must do so with circumspection. With this caution it may prove to be of some value to teachers who do not care to depart entirely from Euclid's theory of proportion, yet find the ordinary presentation beyond the grasp of their pupils.

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*Conf. Cremona, Projective Geometry, p. 22.

NOTES.

A NEW edition of the Annual Register of the Society has been published, containing beside the usual information a complete catalogue of the periodicals contained in the Society's library. Copies of the Register have been mailed to the members and extra copies can be obtained from the Secretary.

The opening (January) number of volume 24 of the *American Journal of Mathematics* contains the following memoirs: "Cyclic subgroups of the simple ternary linear fractional group in a Galois field," L. E. DICKSON; "Curves of triple curvature," by J. G. HARDY; "Primary prime functions in several variables and a generalization of an important theorem of Dedekind," by HARRIS HANCOCK; "The plane cubic curve in relation to the circular points at infinity," by R. A. ROBERTS; "Peirce's linear associative algebra," by H. E. HAWKES; "Groups defined by the orders of two generators and the order of their product," by G. A. MILLER.

The number contains a portrait of BENJAMIN PEIRCE.

THE January number (second series, volume 3, number 2) of the *Annals of Mathematics* contains the following papers: "Some applications of the method of abridged notation," by M. BÔCHER; "On the roots of functions connected by a linear recurrent relation of the second order," by M. B. PORTER; "Space of constant curvature," by F. S. WOODS.

AT THE regular meeting of the London mathematical society on December 12, 1901. Professor A. E. H. LOVE communicated a paper by Mr. J. H. MICHELL on the flexure of plates, and Lieut. Col. A. J. C. CUNNINGHAM gave a sketch of Euler's method of finding amicable numbers and announced two new primes. The preliminary programme for the January meeting of this year consisted of the following papers: "Non-uniform convergence and the integration of series," by the president, Dr. E. W. HOBSON; "Network," by Mr. S. ROBERTS; "On quartic curves with a triple point," by Mr. A. B. BASSET; "On the integrals of the differential equation $du/\sqrt{f(u)} + dv/\sqrt{f(v)} = 0$, where $f(x) \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e$, considered geometrically," by Professor W. SNOW BURNSIDE; "On the fundamental theorem of differential equations," by Mr. W. H. YOUNG.

At a meeting of the Liverpool mathematical society held in University College on Wednesday, December 18, 1901, Professor Sebastian Sircom (President) took the chair, and delivered the third part of his paper "On some hydrodynamical problems examined," dealing principally with the stream lines of a fluid flowing past the cassinian oval. Mrs. A. Boole Stott showed and explained some models and diagrams illustrating rotation in four-dimensional space about a plane; and Mr. H. W. Curjel gave some notes with regard to axes of regular solids and planes of symmetry of regular hypersolids.

Among the prizes awarded by the Paris academy of sciences at its annual meeting, December 16, 1901, are the following:—In geometry: The Francoeur prize to L. Laugel; the Poncelet prize to E. Borel.—In mechanics: The extraordinary prize of six thousand francs divided equally between MM. Tissot and Marbec; the Montyon prize to A. Witz; the Plumey prize to Professor Boulvin of the University of Gand; the Fourneyron prize was not awarded.—In astronomy: The Lalande prize to M. Thome; the Valz prize to M. Ch. André.—The Prix Petit D'Ormoy in the mathematical sciences was awarded to Professor G. Koenigs.

The subjects proposed for the current year include the following:—In geometry: The subject proposed for the grand prize of the mathematical sciences is to perfect, in an important point, the application of the theory of continuous groups to partial differential equations; for the Bordin prize (3,000 francs), to develop and perfect the theory of surfaces applicable to the paraboloid of revolution; the Francoeur prize (1,000 francs) and the Poncelet prize (2,000 francs) will be awarded for works useful to the progress of pure or applied mathematics.—In mechanics: The Plumey prize (2,500 francs) for an improvement in the steam engine or any other invention contributing to the progress of steam navigation; a Montyon prize (700 francs) for invention or improvement of instruments; the extraordinary prize of 6,000 francs for any invention tending to improve the efficacy of the French naval forces.—In astronomy: The Damoiseau prize (1,500 francs) for the completion of the theory of Saturn as given by Leverrier, publishing the rectifying formulæ and establishing the agreement between theory and observation; the Janssen gold medal for an important discovery in physical astronomy; the Lalande prize (540 francs) and the Valz prize (460 francs) for general work in astronomy.

THE PRINCE JABLONOWSKI SOCIETY of Leipsic, announces for 1902 a prize of 1,000 marks for an adequate treatment of the following subject:

"To complete in any essential respect the investigations contained in the paper of Poincaré: 'La méthode de Neumann et le problème de Dirichlet,' *Acta Mathematica*, volume 20 (1896), page 59."

The prize subject proposed by the Society for 1903 is the following:

"By thorough and incontestable experimental investigation, to make an essential contribution to the determination of the laws of light-electric currents."

THE DANISH ACADEMY OF SCIENCES offers its gold medal for a satisfactory discussion of the following question:

"Whether every family of tortuous curves contains limiting forms composed of straight lines." In the case of a negative answer to the general question, the requisite limitation and conditions are to be discussed.

DR. FELIX HAUSDORFF, docent in astronomy and mathematics at the University of Leipsic, has been promoted to an assistant professorship.

DR. FAGUART has been made assistant professor of mathematics at the University of Geneva.

PROFESSOR MIE, of Karlsruhe, has been called to an assistant professorship of mathematical physics at the University of Greifswald.

PROFESSOR N. I. HATZIDAKIS, of the Military Academy at Athens, has been appointed professor of mathematical analysis at the University of Athens.

MR. E. LASKER, the celebrated chess-player, has been made lecturer in mathematics in Owens College, Manchester, England.

PROFESSOR L. MATTHIESSEN, of the University of Rostock, is concluding the one-hundredth semester of his ordinary professorship.

DR. B. O. PEIRCE, Hollis professor of mathematics and natural philosophy at Harvard University, has returned from Europe. He expects to resume the duties of his professorship at the beginning of the next academic year.

DR. C. A. LAISANT, examiner at the Ecole polytechnique and editor of the *Nouvelles Annales de mathématiques*, and other mathematical journals, has recently spent several weeks travelling in the United States.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARZELÀ (C.). Lezioni di calcolo infinitesimale, date nella r. università di Bologna. Vol. I, parte I. Firenze, Le Monnier, 1901. 8vo. 4+436 pp. Fr. 9.00

BAMBERGER (G.). Die Cassinische Fläche. (Diss.) Bern, 1900. 4to. 33 pp., 2 plates.

BARCHI (A.). Sopra una classe di superficie applicabili e sulle loro flessioni; tesi di laurea. Roma, Bertero, 1901. 8vo. 34 pp.

BONNEL (J. F.). La géométrie atomique rationnelle. Paris, Gauthier-Villars, 1902. 8vo. 99 pp. Fr. 5.00

BOUTON (C. L.). Invariants of the linear differential equation and their relation to the theory of continuous groups. (Diss.) Leipzig, 1901. 4to. 62 pp.

CHARLOUIS (M. L.). Handleiding bij de studie der integraalrekening. (In 3 deelen.) 1te gedeelte. Delft, 1901. 8vo. 2+191 pp. Fr. 7.50

COUTURAT (L.). La logique de Leibniz, d'après des documents inédits. Paris, Alcan, 1901. 8vo. 12+608 pp. Fr. 12.00

DURÈGE (H.). Elements of the theory of functions of a complex variable with especial reference to the methods of Riemann. Authorized translation from the fourth German edition by G. E. Fisher and I. J. Schwatt. New York, Macmillan, 1901. 8vo. 13+288 pp. Cloth. $2.00

FISHER (G. E.). See DURÈGE (H.).

GIBSON (G. A.). An elementary treatise on the calculus, with illustrations from geometry, mechanics and physics. London and New York, Macmillan, 1901. 12mo. 19+459 pp. Cloth. $1.90

GOERING (W.). Ueber die Natur und Methode des mathematischen Erkennens. (Festrede.) Dresden, Gewerbe-Buchhandlung, 1901. 8vo. 12 pp. M. 0.50

HAPPEL (H.). Untersuchungen über die Konvergenz der beim Problem der drei Körper auftretenden Reihenentwickelungen. (Diss.) Göttingen, 1900. 8vo. 60 pp.

JAGLARZ (A.). Heron z Aleksandryi i jego problemat powierzchni trójkata. (Progr.) Kraków, 1901. 8vo. 14 pp.

JAMET (V.). Sur les équations anharmoniques. Marseille, Barlatier [1901]. 4to. 21 pp.

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